Random walks and electrical networks 3 + Galton-Watson trees

Thursday, 18 March 2021 12:55

Reminder: $G=(V,E)$ graph, finite connected, $a,z\in V$

$(c_e)_{e\in E}$ positive numbers (conductances) $r_e=\frac{1}{c_e}$

Random walk: $\mathbb{P}(X_{t+1}=v \mid X_t=u)=\dfrac{c_{uv}}{\sum\limits_{w:w\sim u} c_{wu}}$.

directed edges

Flow: $\theta:\vec{E}\to\mathbb{R}$ satisfying node law at any vertex $v\neq a,z$ and is antisymmetric.

$$\|\theta\|:=\sum_{u:u\sim a}\theta_{au} \quad \text{strength}$$

Theorem (Thomson's principle):

$$R_{eff}(a\leftrightarrow z):=\inf\{\mathcal{E}(\theta):\theta \text{ flow}, \|\theta\|=1\}$$

$$\mathcal{E}(\theta):=\frac{1}{2}\sum_{\vec{e}\in\vec{E}} r_{\vec{e}}\,\theta(\vec{e})^2=\sum_{e\in E} r_e\,\theta(e)^2$$

We saw an extension to infinite graphs where we had $R_{eff}(a\leftrightarrow\infty)$.

Connection to random walk:

$$R_{eff}(a\leftrightarrow z)=\frac{1}{\mathbb{P}_a(\underbrace{\tau_z<\tau_a^+}_{\text{get to } z \text{ before returning to } a})\sum\limits_{u:u\sim a} c_{au}}$$

Similarly

$$R_{eff}(a\leftrightarrow\infty)=\frac{1}{\mathbb{P}_a(\text{walk never returns to } a)\sum\limits_{u:u\sim a} c_{au}}.$$

Corollary: Random walk is recurrent iff $R_{eff}(a\leftrightarrow\infty)=\infty$.

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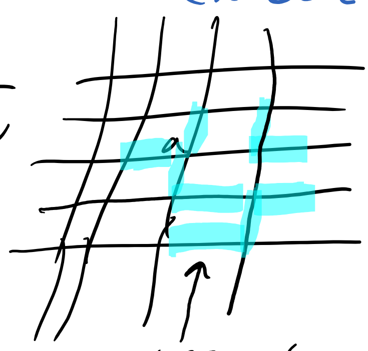
Nash-Williams inequality

A cutset between a and z is a set of edges T separating a from z (every path from a to z must use an edge of T).

Prop.: For every flow from a to z and every cutset T,

$$\sum_{e \in T} |\theta(e)| \geq 1.$$



(Sketch: reduce T to a minimal cutset, then a unit flow must leave the side of a of the cutset towards the side of z).

Thm. (Nash-Williams): Let (T_n) be disjoint cutsets separating a from z. Then

$$R_{eff}(a \leftrightarrow z) \geq \sum_n \Big(\sum_{e \in T_n} c_e \Big)^{-1}.$$

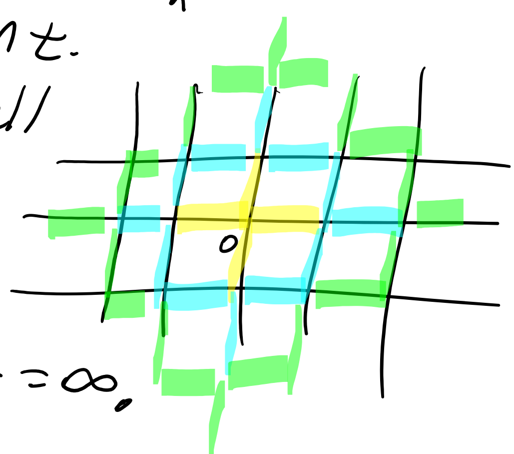
Similarly if $z = \infty$,

$$R_{eff}(a \leftrightarrow \infty) \geq \sum_n \Big(\sum_{e \in T_n} c_e \Big)^{-1}.$$

Example: $\mathbb{Z}^2$ is recurrent.

T_n = boundary of the ball of radius n around 0.

$$|T_n| = \Theta(n).$$



$$\Rightarrow R_{eff}(0 \leftrightarrow \infty) \geq \sum_n \frac{1}{|T_n|} = \infty.$$

$\Rightarrow \mathbb{Z}^2$ is recurrent

$\Rightarrow$ [illegible]

$\Rightarrow \mathbb{Z}^2$ is recurrent.

Proof of theorem: Let θ be a flow with $\|\theta\|=1$.

$$\mathcal{E}(\theta)=\sum_{e\in E} r_e\,\theta(e)^2 \geq \sum_n \sum_{e\in T_n} r_e\,\theta(e)^2.$$

Prop. For a given n,

$$1 \leq \sum_{e\in T_n} |\theta(e)| = \sum_{e\in T_n} \underbrace{\frac{1}{\sqrt{r_e}}}_{\sqrt{c_e}} \sqrt{r_e}\,|\theta(e)| \leq \Big(\sum_{e\in T_n} c_e\Big)^{\frac{1}{2}} \Big(\sum_{e\in T_n} r_e\,\theta(e)^2\Big)^{\frac{1}{2}}$$

$$\Rightarrow \sum_{e\in T_n} r_e\,\theta(e)^2 \geq \frac{1}{\sum_{e\in T_n} c_e}.$$ Proving the thm.

Remark: There are recurrent graphs for which the Nash-Williams does not prove recurrence.

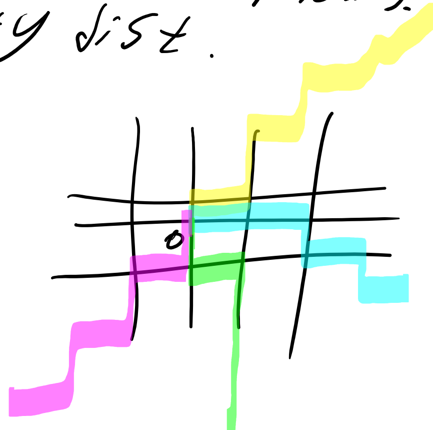
Random paths

This is a method to generate unit flows (it may even generate the unit current flow).

Suppose $\mathbb{P}$ is a probability dist. over paths from a to z.

For a path γ define

$\theta_\gamma(\vec{e})$ = number of times $\vec{e}$ is traversed by γ — number of times $\overleftarrow{e}$ is traversed by γ.



Define $\theta(\vec{e}) = \mathbb{E}\,\theta_\gamma(\vec{e})$ when γ is sampled from $\mathbb{P}$.

Claim: θ is a flow from a to z, $\|\theta\|=1$.

Proof: Check that for each path γ from a to z, θ_γ is a flow of unit strength. This passes to the expected value.

Example: $\mathbb{Z}^3$ is transient.

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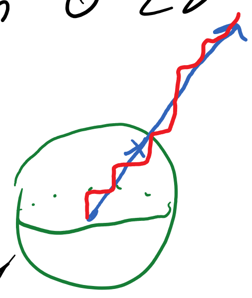
Proof sketch: Given a unit vector $\vec{n} \in S^2 \subseteq \mathbb{R}^3$, let $\gamma_{\vec{n}}$ be a lattice approximation of the straight line from 0 to ∞ in the direction $\vec{n}$.



Let $\theta(\vec{e}) = \mathbb{E}\,\theta_{\gamma_{\vec{n}}}(\vec{e})$ when $\vec{n}$ is chosen uniformly on S^2. **Idea:** $|\theta(\vec{e})| \approx \frac{1}{d(0,\vec{e})^2}$.

$$\Rightarrow \mathcal{E}(\theta) = \sum_{e \in E(\mathbb{Z}^3)} \theta(e)^2 \approx \sum_n n^2 \frac{1}{n^4} < \infty$$

n^2: number of edges in boundary of the ball of radius n around origin

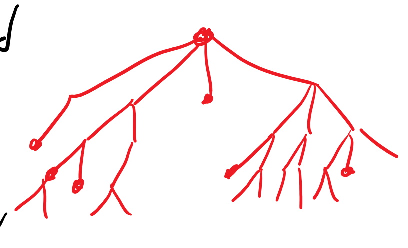
By Thomson's principle, $R_{eff}(0 \leftrightarrow \infty) < \infty$ so $\mathbb{Z}^3$ is transient.

Galton-Watson trees (E.g., Lyons-Peres book)

Background: Francis Galton was interested in the dissapearance of family names. Random model: Say that a person has a random number of children, sampled according to a distribution μ (μ is supported $\{0,1,2,\ldots,\}$).

Say that this indep. from person to person.

Reverend Watson analyzed this in 1874.



Basic theorem - Family tree will die out a.s. iff the average number of children is smaller, or equal, to 1 (unless μ is supp.

almost surely

almost surely children is smaller, or equal, to 1 (unless μ is supp. on $\{1\}$).

Formal statement:

Let μ be supp. on $\{0,1,2,\dots\}$.

We just discuss the number of children at every generation instead of the full tree.

Let $Z_0 := 1$. For each $n \geq 1$,

let $Z_{n+1} := \sum_{k=1}^{Z_n} X_n^{(k)}$ ← number of children of k'th person in generation n

number of people in generation $n+1$

where $(X_n^k)_{\substack{n\geq 0 \\ k \geq 1}}$ are indep. samples of μ.

Basic question: is the survival probability $P(\forall n, Z_n > 0) > 0$?

Theorem: $P(\forall n, Z_n > 0) > 0$ iff

$m := \mathbb{E}X > 1$ or $P(X=1)=1$ (average number of children)

where $X \sim \mu$.

Example: percolation on a binary tree.

Each edge is kept with prob. $p \in [0,1]$, indep. among edges.

$$P(\exists \text{ an infinite conn. comp.}) = \begin{cases} 0 & p \leq \frac{1}{2} \\ 1 & p > \frac{1}{2} \end{cases}$$

$\{0,1\} \ni$ by Kolmogorov's 0-1 law

Conn. comp. of root is a GW tree with $\mu = Bin(2,p)$.

Proof of thm.: The case $m < 1$:

Note that $\mathbb{E}Z_n = \mathbb{E}(\mathbb{E}(Z_n | Z_{n-1})) =$ $(n \geq 1)$

$$= \mathbb{E}\left(\mathbb{E}\left(\sum_{k=1}^{Z_{n-1}} X_{n-1}^{(k)} \,\middle|\, Z_{n-1}\right)\right) = \mathbb{E}(m Z_{n-1})$$

$$- \quad = m^n. \qquad P(Z_n \geq 1)$$

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By Markov's ineq., $P(Z_n > 0) \leq m^n$

Note $\{\forall n \; Z_n > 0\} = \bigcap_n \{Z_n > 0\} \Rightarrow P(\forall n, Z_n > 0) =$

$$= \lim_{n\to\infty} P(Z_n > 0) \xrightarrow[m<1]{} 0$$

The case $m=1$:

Note that, for any finite m,

$$M_n := \frac{Z_n}{m^n} \text{ is a martingale}$$

Indeed, $\mathbb{E}(M_n \mid Z_0, \ldots, Z_{n-1}) = \qquad (n \geq 1)$

$$= \frac{1}{m^n}\mathbb{E}(Z_n \mid Z_{n-1}) = \frac{1}{m^n} m Z_{n-1} = M_{n-1}$$

In particular, when $m=1$, Z_n is a mart.

By the mart. conv. theorem $(Z_n \geq 0)$

finite almost surely $\exists Z_\infty$ s.t. $Z_n \to Z_\infty$ almost surely.

Since Z_n is integer-valued we conclude that, a.s., (Z_n) is eventually constant.

If $P(X=1) \neq 1$ then this is only possible if (Z_n) is eventually zero.

The general case:

We may assume that $\mathbb{E}X < \infty$ since otherwise we may truncate X (replace it by $\min(X, M)$ for M large enough) and analyze $P(\forall n, Z_n > 0)$ for the truncated process.

Probability generating function:

$$f(s) := \mathbb{E}(s^X) \quad \text{for } 0 \leq s \leq 1$$

with $f(0) := P(X=0)$, $f(1) := 1$.

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$$f(s) = \sum_{k=0}^{\infty} P(X=k)\, s^k.$$

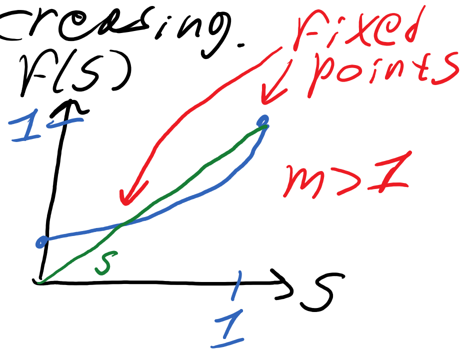
Properties: i) f is non-decreasing.

ii) f is convex.

iii) $f'(1) = \mathbb{E}(X) = m$.

left derivative

In fact, $f'(s) = \mathbb{E}(X s^{X-1})$



Define $f_n(s) := \mathbb{E}(s^{Z_n})$ so that $f_1 = f$.

Claim: $f_{n+1}(s) = f(f_n(s))$ for $s \in [0,1]$.

Proof: $f_{n+1}(s) = \mathbb{E}(s^{Z_{n+1}}) = \mathbb{E}(\mathbb{E}(s^{Z_{n+1}} \mid Z_n)) =$

$$= \mathbb{E}\Big(\prod_{k=1}^{Z_n} \mathbb{E}(s^X)\Big) = \mathbb{E}\Big(\prod_{k=1}^{Z_n} f(s)\Big) =$$

$$Z_{n+1} = \sum_{k=1}^{Z_n} X_n^{(k)}$$

cond. indep. given Z_n and dist. μ

$$= \mathbb{E}(f(s)^{Z_n}) = f_n(f(s))$$

by def.

Thus $f_n(s) = \underbrace{f(f(f(\dots f(s)))\dots)}_{n \text{ compositions}}$

By def. $P(Z_n = 0) = f_n(0)$

by claim $\longrightarrow = f(f_{n-1}(0)) = f(P(Z_{n-1}=0))$

In particular, $P(Z_n=0) \longrightarrow P\binom{\text{process}}{\text{dies out}} =: q$

extinction prob.

Thus we would like to investigate $f(f(f(\dots(0)))\dots)$ as we apply f more and more.

f is convex, hence continuous,

$P(Z_n=0) = f(P(Z_{n-1}=0))$

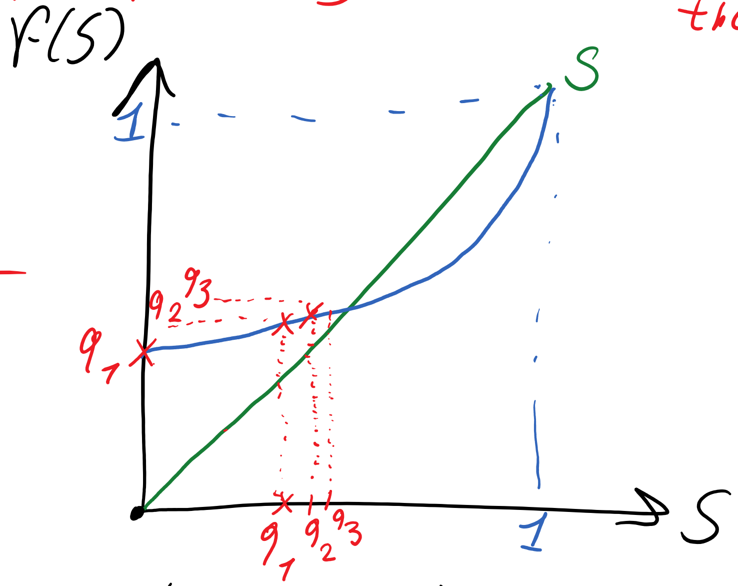
f is convex, hence it continuous,

So since $P(Z_n=0) = f(P(Z_{n-1}=0))$

$$\boxed{q = f(q)}$$

Extinction prob. is a fixed point of prob. gen. fcn. In fact, it is the smallest fixed point.

m > 1:

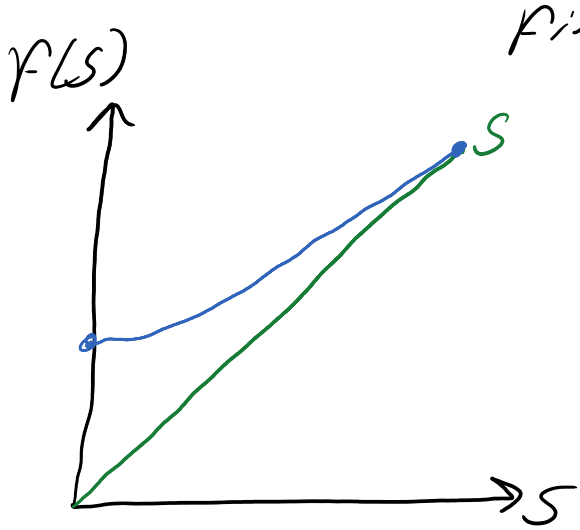


Thus $P(\forall n, Z_n > 0) > 0$

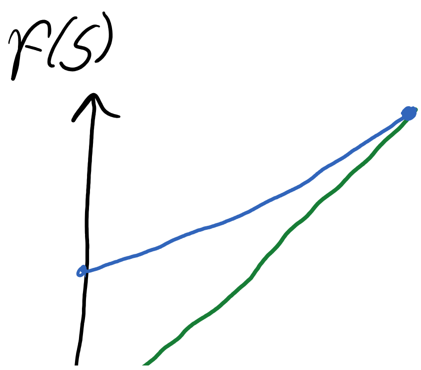
$\qquad\qquad\quad \overset{\shortparallel}{1-q}$ where q is the smallest fixed point of f.

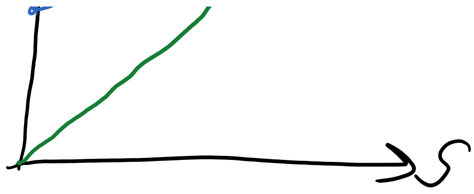
m = 1:

$P(X=1) \neq 1$



m < 1:





Sub-critical case: $m<1$.

We saw that $\mathbb{P}(Z_n>0)\le m^n$.
How sharp is this?

Theorem (Heathcote, Seneta–Jones 1967):

$\forall m$, $\dfrac{\mathbb{P}(Z_n>0)}{m^n}$ is non-increasing and thus converges.
For $m<1$, the limit is positive iff $\mathbb{E}(X\log X)<\infty$ where $0\log(0):=0$.

Super-critical case: $m>1$.

Notice that $\mathbb{E}(Z_n)=m^n$. Is Z_n really of order m^n for all n on the event of survival?
Recall that $M_n=\dfrac{Z_n}{m^n}$ is a non-neg. martingale.
So $M_n\to M_\infty$ a.s. Is $M_\infty>0$ a.s. on survival?

Thm. (Kesten–Stigum 1966): When $m>1$ the following are equivalent:
i) $\mathbb{P}(M_\infty>0)=\mathbb{P}(\forall n, Z_n>0)$.
ii) $\mathbb{E}(M_\infty)=1$.
iii) $\mathbb{E}(X\log X)<\infty$.
If these cond. are violated, $M_\infty=0$ a.s.

It is easy to see that the cond. in the Kesten-Stigum thm. hold if $\mathbb{E}(X^2)<\infty$.
This is the case since

hold if ...

This is the case since

$$Var(Z_n) = var(X)\begin{cases} \frac{m^n(m^n-1)}{m^2-m} & m \neq 1 \\ n & m=1 \end{cases}$$

E.g., by cond. on Z_{n-1} and using the total variance formula.

Thus, when $m>1$, $\sup_n \mathbb{E}(M_n^2)<\infty$.

Hence (M_n) is a martingale bounded in L^2

So $M_n \to M_\infty$ in L^2.

It follows that $\mathbb{E}(M_\infty)=1$.

In addition,

$$P(M_\infty=0) = \mathbb{E}(P(M_\infty=0 / Z_1)) =$$

$$= \mathbb{E}(P(M_\infty=0)^{Z_1}) = f(P(M_\infty=0)).$$

So $P(M_\infty=0)$ is a fixed point of f.

Combined with $\mathbb{E}M_\infty=1 \Rightarrow P(M_\infty>0)=$

$$=P(\forall n, Z_n>0).$$